Chapter 1

Canonical Froms of 2nd Order Linear PDE's

Consider the PDE

$$a\,u_{xx} + \mathbf{2}b\,u_{xy} + c\,u_{yy} + d\,u_x + e\,u_y + f\,u + g = 0$$

where coefficients $a, \cdots, g$ are functions of x and y and $u = u(x,\, y)$.

- This PDE is called **hyperbolic** if $b^2 > ac$ or $b^2 - ac > 0$. The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$ is a hyperbolic equation.
- This PDE is called **parabolic** if $b^2 = ac$ or $b^2 - ac = 0$. The heat equation $\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$ is a parabolic equation.
- This PDE is called **elliptic** if $b^2 < ac$ or $b^2 - ac < 0$. The Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is an elliptic equation.

By an appropriate change of variables the PDE $a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} + d\,u_x + e\,u_y + f\,u + g = 0$ can be written in its canonical form. The canonical forms are

- **Hyperbolic** PDE: $u_{\xi\eta}$ + Lower Ordered Terms $= 0$. Incorporating a 45 degree rotation of coordinate sytem into the change of variables, this equation can be written in the form $u_{\xi\xi} - u_{\eta\eta}$ + Lower Ordered Terms $= 0$.
- **Parabolic** PDE: $u_{\xi\xi}$ + Lower Ordered Terms $= 0$.
- **Elliptic** PDE: $u_{\xi\xi+}u_{\eta\eta}$ + Lower Ordered Terms $= 0$.

The characteristic equation of the original PDE is the differential equation

$$a\left(\frac{dy}{dx}\right)^2 - 2b\,\frac{dy}{dx} + c = 0$$

Notice that the chorarcteric equation has $-2b$ as a coefficient, while the PDE has $+2b$.

- In the case of a hyperbolic PDE, the characteristic equation has two real and unequal solutions

$$\frac{dy}{dx} = \frac{-(-2b) \pm \sqrt{(-2b)^2 - 4ac}}{2a} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

Solve these ODE's for y, and write the solutions in the form

$$\phi(x,\, y) = c_1 \text{ and } \psi(x,\, y) = c_2$$

The change of variable

$$\xi = \phi(x,\, y)\,,\ \ \eta = \psi(x,\, y)$$

will reduce the PDE to its canonical form.

- In the case of a parabolic PDE, the characteristic equation has one solution

$$\frac{dy}{dx} = \frac{-(-2b)}{2a} = \frac{b}{a}$$

Solve this ODE for y, and write the solution in the form

$$\phi(x,\, y) = c_1$$

Let $\xi = \phi(x,\, y)$ and choose smooth function $\eta = \psi(x,\, y)$ independent of ξ; choose η so that the Jacobian $J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$.

The change of variable

$$\xi = \phi(x,\, y)\,,\ \ \eta = \psi(x,\, y)$$

will reduce the PDE to its canonical form.

- In the case of an elliptic PDE, the characteristic equation has two complex conjuage solutions

$$\frac{dy}{dx} = \frac{-(-2b) \pm \sqrt{(-2b)^2 - 4ac}}{2a} = \frac{b}{a} \pm i\,\frac{\sqrt{ac - b^2}}{a}$$

Choose one of the ODE's, say $\frac{dy}{dx} = \frac{b}{a} + i\,\frac{\sqrt{ac-b^2}}{a}$, solve it for y, and write the solution in the form

$$\phi(x,\, y) + i\,\psi(x,\, y) = c_1$$

The change of variable

$$\xi = \phi(x,\, y)\,,\ \ \eta = \psi(x,\, y)$$

will reduce the PDE to its canonical form.